Model Solutions

Additional Assessment Materials

Summer 2021

Pearson Edexcel GCE in Mathematics

9MA0 (Public release version)

Resource Set 1: Topic 7

Differentiation (Test 2)

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Additional Assessment Materials, Summer 2021

General guidance to Additional Assessment Materials for use in 2021

Context

- Additional Assessment Materials are being produced for GCSE, AS and A levels (with the exception of Art and Design).
- The Additional Assessment Materials presented in this booklet are an optional part of the range of evidence teachers may use when deciding on a candidate's grade.
- 2021 Additional Assessment Materials have been drawn from previous examination materials, namely past papers.
- Additional Assessment Materials have come from past papers both published (those materials available publicly) and unpublished (those currently under padlock to our centres) presented in a different format to allow teachers to adapt them for use with candidate.

Purpose

- The purpose of this resource to provide qualification-specific sets/groups of questions covering the knowledge, skills and understanding relevant to this Pearson qualification.
- This document should be used in conjunction with the mapping guidance which will map content and/or skills covered within each set of questions.
- These materials are only intended to support the summer 2021 series.

1. Given $y = x(2x + 1)^4$, show that

$$\frac{dy}{dx} = (2x + 1)^n (Ax + B)$$

where n, A and B are constants to be found.

We use the product r Since $y = f(x) \cdot g(x)$ with $f(x) = x \to f'(x) = 1$

$g(x) = (2x+1)^4 \to 4 \times 2(2x+1)^3 = 8(2x+1)^3$

multiply by the Power and the derivative of inside the bracket.

Then $\frac{dy}{dx} = f'(x) \cdot g(x) + f(x) \cdot g(x)'$

$= (2x+1)^4 + 8x(2x+1)^3$

$\Rightarrow \frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$

$= (2x+1)^3((2x+1) + 8x)$ take a common factor of $(2x+1)^3$ out.

$\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n = 3,\ A = 10$ and $B = 1$

(Total for Question 1 is 4 marks)

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2. A curve C has equation

$$y = x^2 - 2x - 24\sqrt{x}, \quad x > 0.$$

(a) Find (i) $\frac{dy}{dx}$, $\frac{dy}{dx} = 2x - 2 - \frac{12}{\sqrt{x}}$

$-24\sqrt{x} = -24x^{1/2}$

$= -12x^{-1/2}$

$= \frac{-12}{\sqrt{x}}$

(ii) $\frac{d^2y}{dx^2}$, $\frac{d^2y}{dx^2} = 2 - \frac{6}{x^{3/2}}$

$-12\sqrt{x} = -12x^{-1/2}$

$= 6x^{-3/2}$

$= \frac{6}{x^{3/2}}$ **(3)**

(b) Verify that C has a stationary point when $x = 4$.

Stationary point when $\frac{dy}{dx} = 0$ when $x = 4$. **(2)**

$\Rightarrow \frac{dy}{dx} = 2x - 2 - \frac{12}{\sqrt{x}}$ evaluated at 4: $\frac{dy}{dx} = 4 \times 2 - 2 - \frac{12}{\sqrt{4}} = 0$ as required

(c) Determine the nature of this stationary point, giving a reason for your answer.

Nature of stationary point can be found by finding the value of $\frac{d^2y}{dx^2}$ at x. **(2)**

(Total for Question 2 is 7 marks)

For $x = 4$, $\frac{d^2y}{dx^2} = 2 - \frac{6}{x^{3/2}} = 2 - \frac{6}{4^{3/2}} = \frac{5}{4} > 0$

$\Rightarrow$ Minimum stationary point.

$> 0 \Rightarrow$ minimum

$< 0 \Rightarrow$ maximum

3. $$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}.$$

(a) Find the values of the constants A, B and C.

(4)

$1-2x=0$

$1=2x$

$x=1/2$

$1+11x-6x^2 = A(x-3)(1-2x)+B(1-2x)+C(x-3)$

$x=3 \Rightarrow -20=-5B \Rightarrow \underline{B=4}$

$x=\frac{1}{2} \Rightarrow 5 = -\frac{5}{2}C \Rightarrow \underline{C=-2}$

$x=0 \Rightarrow 1 = -3A+4+6 \Rightarrow 3A = 6+4-1 \Rightarrow A=\frac{9}{3} \Rightarrow \underline{A=3}$

$\Rightarrow \underline{A=3}, \underline{B=4}, \underline{C=-2}$

$$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)}, \quad x>3.$$

(b) Prove that f (x) is a decreasing function.

(3)

If $f'(x)<0$ then $f(x)$ will be a decreasing function.

$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)} = 3+\frac{4}{(x-3)}-\frac{2}{1-2x}$ then $f'(x) = -\frac{4}{(x-3)^2}-\frac{4}{(1-2x)^2}$

$= \underline{-4\left(\frac{1}{(x-3)^2}+\frac{1}{(1-2x)^2}\right)}$

Then we see that on $x>3$, $f'(x)<0$ always, which can clearly be seen by taking the -4 out.

(Total for Question 3 is 7 marks)

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4. $g(x) = 4x^3 + ax^2 + 4x + b$, where a and b are constants.

Given that $(2x + 1)$ is a factor of $g(x)$ and that the curve with equation $y = g(x)$ has a point of inflection at $x = \frac{1}{6}$,

(a) find the value of a and the value of b.

(5)

$2x+1$ is a factor $\Rightarrow x=-\frac{1}{2}$ is a root. $\Rightarrow g(-\frac{1}{2})=0$.

$f(-\frac{1}{2}) = 4(-\frac{1}{2})^3 + a(-\frac{1}{2})^2 + 4(-\frac{1}{2}) + b = 0$ and $g''(\frac{1}{6})=0$

$\Rightarrow -\frac{5}{2}+\frac{1}{4}a+b=0$

$\underline{a+4b=10}$

$\Rightarrow g'(x) = 12x^2+2ax+4$

$g''(x) = 24x+2a$

$g''(\frac{1}{6}) = 4+2a=0$

$\Rightarrow a = -\frac{4}{2} = \underline{-2} \Rightarrow b = \frac{10-(-2)}{4} = \underline{3}$

$\Rightarrow \underline{a=-2}$ and $\underline{b=3}$

(b) Show that there are no stationary points on the curve with equation $y = \mathrm{g}(x)$.

(2)

$g'(x) = 12x^2 - 4x + 4 = 0$

$\Rightarrow 3x^2 - x + 1 = 0$ but $b^2 - 4ac = 1 - 4(3)(1) = -11 < 0$

$\Rightarrow$ No roots exist

$\Rightarrow$ No stationary points

(Total for Question 4 is 7 marks)

5.

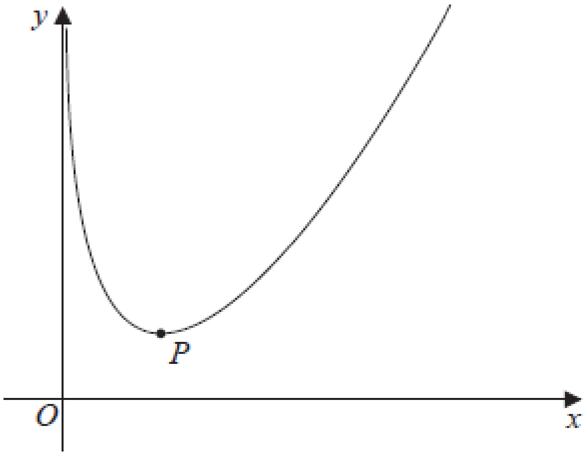

Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \ , \quad x > 0$$

(*a*) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

let $h(x) = \frac{4x^2+x}{2\sqrt{x}}$ and we will find $h'(x)$ using the quotient Rule.

$f(x) = 4x^2+x \rightarrow f'(x) = 8x+1$

$g(x) = 2\sqrt{x} \rightarrow g'(x) = \frac{1}{\sqrt{x}}$

$(g(x))^2 = 4x$

$\Rightarrow h'(x) = \dfrac{(8x+1)\,2\sqrt{x} - \frac{4x^2+x}{\sqrt{x}}}{4x}$

$h'(x) = \dfrac{16x^{3/2} + 2\sqrt{x} - 4x^{3/2} - \sqrt{x}}{4x} = \dfrac{12x^{3/2} + \sqrt{x}}{4x}$

Then let $m(x) = -4\ln x$ then $m'(x) = -\frac{4}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{12x^{3/2} + \sqrt{x}}{4x} - \dfrac{4}{x} = \dfrac{12x^{3/2} + \sqrt{x}}{4x} - \dfrac{16}{4x} = \dfrac{12x^{3/2} + \sqrt{x} - 16}{4x}$

take common factor of $\frac{1}{\sqrt{x}}$ out.

$= \dfrac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4\sqrt{x}x} = 0$$

$$\Rightarrow \quad 12x^2 + x - 16\sqrt{x} = 0$$

$$\Rightarrow \quad x^{1/2}(12x^{3/2} + x^{1/2} - 16) = 0$$

$$\Rightarrow \quad 12x^{3/2} + x^{1/2} - 16 = 0$$

$$\Rightarrow \quad 12x^{3/2} = 16 - x^{1/2}$$

$$\Rightarrow \quad x^{3/2} = \frac{4}{3} - \frac{x^{1/2}}{12}$$

$$x = \left(\frac{4}{3} - \frac{x^{1/2}}{12}\right)^{2/3} \quad \text{as required.}$$

$\Rightarrow$ x is <u>a solution.</u>

(Total for Question 5 is 7 marks)

6.

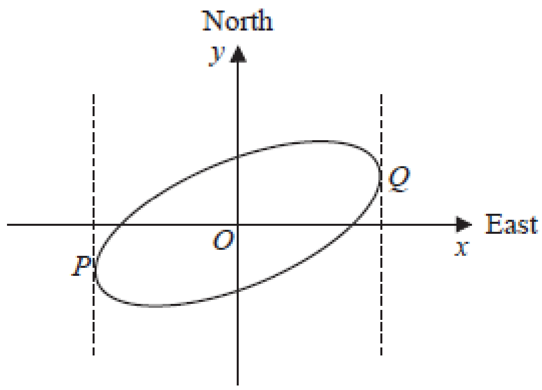

Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$.

(4)

$x^2 - 2xy + 3y^2 = 50$

$2x - 2y - 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-2x + 6y) = 2y - 2x \Rightarrow \frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} = \frac{y - x}{3y - x}$ as Required.

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P.

think of rotating the graph by 90°.

(5)

P is a point which has a vertical tangent which means that $\frac{dx}{dy} = 0$.

$\Rightarrow \frac{3y - x}{y - x} = 0 \Rightarrow x = 3y \Rightarrow$ we can sub this back into the original equation.

$(3y)^2 - 2(3y)y + 3y^2 = 50 \Rightarrow 9y^2 - 6y^2 + 3y^2 = 50 \Rightarrow 6y^2 = 50 \Rightarrow y = \pm\sqrt{\frac{50}{6}} = \pm\frac{5\sqrt{3}}{3}$

we identify graphically that the y coordinate of P will be $-\frac{5\sqrt{3}}{3} \Rightarrow x = 3 \times -\frac{5\sqrt{3}}{3} = -5\sqrt{3}$

$\Rightarrow P = \left(-5\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)$

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O. (You **do not** need to carry out this calculation).

We should set $\frac{dy}{dx}$ equal to zero $\left(\frac{dy}{dx} = 0\right) \Rightarrow \frac{y-x}{3y-x} = 0 \Rightarrow y = x$. **(1)**

Then sub $y = x$ into $x^2 - 2xy + 3y^2 = 50$ and solve for x.

(Total for Question 6 is 10 marks)

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7. A company decides to manufacture a soft drinks can with a capacity of 500 ml.

The company models the can in the shape of a right circular cylinder with radius r cm and height h cm. In the model they assume that the can is made from a metal of negligible thickness.

(a) Prove that the total surface area, S cm^2, of the can is given by

$$S = 2\pi r^2 + \frac{1000}{r}$$

$S = 2\pi rh + 2\pi r^2$ **(3)**

and $V = \pi r^2 h$

$500 = \pi r^2 h$

$\frac{1000}{r} = \frac{2\pi r^2 h}{r} = 2\pi rh \Rightarrow S = \frac{1000}{r} + 2\pi r^2$ as required.

Given that r can vary,

(b) find the dimensions of a can that has minimum surface area.

(5)

$\frac{dS}{dr} = -\frac{1000}{r^2} + 4\pi r = 0 \Rightarrow 4\pi r^3 = 1000$

$r = \left(\frac{1000}{4\pi}\right)^{1/3} = 4.3$ cm

Then h: $500 = \pi r^2 h \Rightarrow h = \frac{500}{\pi (4.3)^2} = 8.6$ cm $\Rightarrow$ height $= 8.6$ cm, radius $= 4.3$ cm.

(c) With reference to the shape of the can, suggest a reason why the company may choose not to manufacture a can with minimum surface area.

(1)

A can with minimum surface area might be a unique shape which could effect how it looks / feels in the hand which could have a negative effect on sales.

(Total for Question 7 is 9 marks)

8.

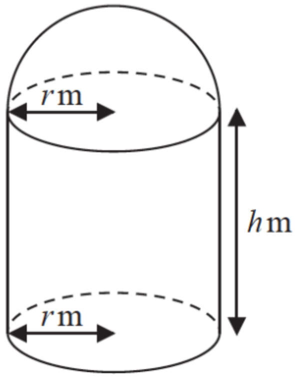

Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is 6 m^3.

(*a*) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

Our total surface area will be the surface area of:

- Circular bottom of cyclinder/base $= \pi r^2$
- Curved surface of cyclinder $= 2\pi rh$ $\Rightarrow S = 3\pi r^2 + 2\pi rh$
- Curved surface of hemisphere $= 2\pi r^2$

Then $V = \pi r^2 h + \frac{1}{2} \times \frac{4}{3}\pi r^3 = 6 \Rightarrow \frac{6 - \frac{2}{3}\pi r^3}{\pi r^2} = h$

cyclinder

half a circle, i.e. hemisphere

Then $S = 3\pi r^2 + 2\pi r\left(\frac{6 - \frac{2}{3}\pi r^3}{\pi r^2}\right)$

$$S = 3\pi r^2 + \frac{12\pi r - \frac{4}{3}\pi^2 r^4}{\pi r^2} = 3\pi r^2 + \frac{12\pi r}{\pi r^2} - \frac{4}{3}\frac{\pi^2 r^4}{\pi r^2}$$

$$= 3\pi r^2 + \frac{12}{r} - \frac{4}{3}\pi r^2$$

$\Rightarrow S = \frac{12}{r} + \frac{5}{3}\pi r^2$ as required

The manufacturer needs to minimise the surface area of the tank.

(*b*) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

$$\frac{dS}{dr} = -\frac{12}{r^2} + \frac{10}{3}\pi r = 0$$

$$\Rightarrow \quad 36 = 10\pi r^3$$

$$r = \sqrt[3]{\frac{36}{10\pi}} = \underline{\underline{1.05 \text{ m}}}$$

(*c*) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)

$$S = \frac{12}{r} + \frac{5}{3}\pi r^2 = \frac{12}{1.05} + \pi (1.05)^2 = 17.201\ldots = \underline{\underline{17 \text{ m}^2}}$$

(Total for Question 8 is 10 marks)